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The Saint-Venant torsion of a circular bar consisting of a composite cylinder assemblage with cylindrically orthotropic constituents

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Abstract

We consider the Saint-Venant torsion of a cylindrical rod of a circular cross section which is filled up by an assemblage of composite circular cylinders. The constituent cylinders consist of a core and a coating both of which are cylindrically orthotropic with the volume fraction of the core being the same in every composite cylinder. The described microstructure is the composite cylinder assemblage of Hashin and Rosen [J. Appl. Mech. 29 (1964) 143] which is now subjected to torsion. The main results are (a) the warping function on the lateral surface of the host rod is zero, (b) an exact expression for the torsional rigidity of the host rod is derived which depends on the size distribution of the composite cylinders but not on their position and (c) there are two circumstances in which the torsional rigidity becomes size distribution independent: The first one is that in which the sizes of the composite cylinders are much smaller than the size of the host rod; the second one is that in which a certain specific relation holds between the properties of the composite cylinder and the volume fraction of the core. If the coating disappears and the core is cylindrically orthotropic, we get the configuration of a polycrystalline rod. Simple bounds for the torsional rigidity of the constructed composite rod are obtained.

Keywords: Saint-Venant torsion; Fiber-reinforced composite material; Composite cylinder assemblage

1. Introduction

Although the Saint-Venant torsion problem of cylindrical bars is a classical one in the field of elasticity, there has been recently a growing interest in it specially in the context of inhomogeneous and/or anisotropic bars (see for example, Nazarov, 1995; Rooney and Ferrari, 1995; Horgan and Chan, 1999; Lipton, 1998; Ting, 1999; Chen, 2001; Tarn, 2001; Wineman, 2001; Benveniste and Chen, 2001; Chen et al., 2002). The

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present paper is a contribution to the existing exact benchmark solutions of inhomogeneous cylindrical bars. The considered microgeometry in the cylindrical bar is the renowned composite cylinder assemblage (CCA) of Hashin and Rosen (1964). This microgeometry was aimed at modeling the effective behaviour of fiber reinforced composites and served also to show the realizability of some of the Hill (1964) and Hashin (1964) bounds for fibrous composites. It consists of two phase composite cylinders of circular cross sections and of all sizes which fill up the whole space, with the volume fraction of the core being the same in all composite cylinders. The original formulation assumed that every composite cylinder consists of an isotropic core and coating, but was later extended to the case of transversely isotropic constituents (Hashin, 1972, 1979), as well to constituents with cylindrical orthotropy (Hashin, 1990). For a comprehensive survey of this microgeometry and several generalizations of it, see Chapter 7 in the recent book of Milton (2002).

In the circular rod considered here which is filled up with the CCA microgeometry each composite cylinder consists of a core and coating which are cylindrically orthotropic. This highly inhomogeneous rod is now subjected to Saint-Venant torsion and the resulting displacement field in it as well as its torsional rigidity are sought. The solution to this problem for the case in which the core and coating are isotropic has been recently given by the authors, Chen et al. (2002). That solution was part of an extensive study in which the existence of so-called "neutral inhomogeneities" was explored in torsion problems. A neutral inhomogeneity was defined in that work as one which does not disturb the vanishing warping field in a host circular bar in torsion, and possibly leaves its torsional rigidity unchanged as well. Here we concentrate exclusively on a cylindrical bar with the CCA microgeometry in torsion, and generalize the solution in Chen et al. (2002) to the case in which the constituents are cylindrically orthotropic. We find it remarkable that although the CCA microgeometry is now almost half a century old, to the best knowledge of the authors, these solutions to the Saint-Venant torsion of a cylindrical bar filled up with this microgeometry have not been given in the literature until now.

The paper is structured as follows: in Section 2 we present a brief summary of our main results. These are (a) the warping displacement vanishes at the lateral surface of a cylindrical and circular rod which is filled up with the CCA microgeometry, (b) the torsional rigidity of this bar is given by a simple expression which is independent on the position of the composite cylinders but is dependent on their size distribution and (c) there are two circumstances in which the torsional rigidity becomes size distribution independent: The first one is that in which the sizes of the composite cylinders are much smaller than the size of the host rod; the second one is that in which a certain specific relation holds between the properties of the composite cylinder and the volume fraction of the core. In Section 3 we present the derivation which is analytical and straightforward. This section contains also the proof of a correspondence between the Saint-Venant torsion problem and an anti-plane shear problem of the same geometry, a correspondence which is valid only for the case of a circular cylindrical inhomogeneity in a circular rod. In Section 4 we study in detail the torsional rigidity of the rod which is filled up with the CCA microgeometry. As indicated above, since the torsional rigidity is dependent on the size distribution of the constituent composite cylinders, the question of possible size distributions which result in a maximum or minimum torsional rigidity becomes a relevant one. We show that some simple answers to this question exist in certain circumstances.

Our concern in the present paper has been to provide an exact analysis to the Saint-Venant torsion a cylindrical bar with a circular cross section which is made up of a well-known microgeometry. The analysis which is characterized by its simplicity leads to several unexpected conclusions. Some applications of the derived results can be contemplated in the field of composites. Clearly, if the microgeometry is such that the sizes of the composite cylinders are much smaller than the size of the host rod, the composite bar behaves in a quasi-homogeneous manner with the effective longitudinal shear modulus of the CCA microgeometry. Thus, the contribution of the analysis in this paper concerns circumstances in which some of the composite cylinders which fill up the host rod are of sizes which cannot be considered to be small with the respect to the host bar. Potential applications in the field of carbon–carbon composites (Christensen, 1994; Hashin, 1990; Herakovich, 1989) or bone mechanics (Guo, 2001; Lakes, 1995) may be envisaged.

2. Statement of results

We consider a cylindrical bar of circular cross section which is filled up by composite cylinders of all sizes, see Fig. 1a. Each composite cylinder consists of a core and a coating which are cylindrically orthotropic. The constitutive relations and the volume fraction of the core are the same in each composite cylinder. This microgeometry is the CCA of Hashin and Rosen (1964). The circular rod consisting of this microgeometry is now subjected to Saint-Venant torsion. We are interested in obtaining the displacement fields and the torsional rigidity of this cylinder. A concise summary of the main results will be given in this section and their derivation will be presented in Section 3.

We start by considering a homogeneous isotropic cylindrical bar of length L and of a circular cross section of radius R. The shear modulus of the bar is denoted by $\mu^{(m)}$. Locate a Cartesian coordinate system (X_1, X_2, X_3) centered at one end of the bar and let its origin coincide with the center of the circular cross section. Let X_3 be the axial coordinate. The bar is subjected to end torsional moments which result in the following displacement field in it:

$$u_1 = -\vartheta X_3 X_2, \qquad u_2 = \vartheta X_3 X_1, \qquad u_3 = 0, \tag{2.1}$$

where ϑ is the angle of twist per unit length. Let us now introduce a composite cylinder in the bar. The radius of the core is denoted by b, and the outer radius of the coating by a. Let the axis of the composite cylinder be positioned at $X_1 = a_0$, $X_2 = 0$. At the center of the composite cylinder define a Cartesian coordinate system (x_1, x_2, x_3) and a polar coordinate system (r, θ, x_3), see Fig. 1b. The core and the coating are cylindrically orthotropic with their constitutive law being given by

$$\sigma_{3r}^{(\alpha)} = 2\mu_r^{(\alpha)}\varepsilon_{3r}^{(\alpha)}, \quad \sigma_{3\theta}^{(\alpha)} = 2\mu_\theta^{(\alpha)}\varepsilon_{3\theta}^{(\alpha)}, \qquad \alpha = 1, 2, \tag{2.2}$$

where $\alpha = 1$ denotes the coating and $\alpha = 2$ the core; $(\sigma_{3r}, \sigma_{3\theta})$ are the shear stresses active in Saint-Venant torsion; and $(\varepsilon_{3r}, \varepsilon_{3\theta})$ are the corresponding strains; $(\mu_r^{(\alpha)}, \mu_\theta^{(\alpha)})$ denote the shear moduli characterizing cylindrical orthotropy. We ask the following question: "for given values of a_0, a, b, $\mu_r^{(\alpha)}$, $\mu_\theta^{(\alpha)}$, is there a specific shear modulus $\tilde{\mu}_m$ of the host bar for which the displacement field in it remains undisturbed and continues to be given by (2.1)?" It turns out that the answer to this question is positive and consists of

$$\mu^{(m)} = \tilde{\mu}_m = \sqrt{\mu_r^{(1)}\mu_\theta^{(1)}}\,\frac{g(1 + c^{k_1}) + (1 - c^{k_1})}{g(1 - c^{k_1}) + (1 + c^{k_1})}, \tag{2.3}$$

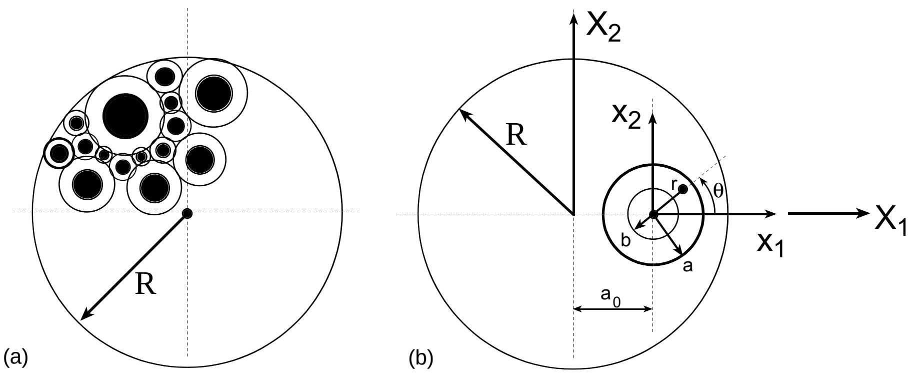

Fig. 1. (a) The composite cylinder assemblage microgeometry. (b) A coated circular inhomogeneity in a circular bar.

where

$$g = \frac{\sqrt{\mu_r^{(2)}\mu_\theta^{(2)}}}{\sqrt{\mu_r^{(1)}\mu_\theta^{(1)}}}, \qquad c = b^2/a^2, \qquad k_1 = \sqrt{\mu_\theta^{(1)}/\mu_r^{(1)}}. \tag{2.4}$$

Remarkably so, this result is independent on the location a_0 of the cylindrical inhomogeneity. Surprisingly as well, it coincides with that given by Hashin (1990) for the effective longitudinal shear modulus of the CCA microgeometry. The reason for this coincidence will be explained in Section 3 which is mainly devoted to the derivation of (2.3). From the nature of the solution, the warping field vanishes at the outer boundary $r = a$ of the coated inhomogeneity. Such inhomogeneities which leave the displacement field undisturbed in the host bar are called "partially neutral inhomogeneities". A "fully neutral inhomogeneity" is one that leaves the torsional rigidity of the host bar undisturbed as well, see Chen et al. (2002).

In Section 3 it is shown that the torsional rigidity of the bar which contains now the partially neutral inhomogeneity is given by

$$T/\vartheta = (\pi/2)\{\tilde{\mu}_m(R^4 - a^4) + \mu_\theta^{(1)}a^4(1 - c^2) + \mu_\theta^{(2)}a^4c^2\}. \tag{2.5}$$

Again, most remarkably so, this expression is independent of the location of the composite cylinder in the host bar!

The fact that (2.3) is independent of a_0 allows us to introduce several composite cylinders in the host bar at arbitrary locations without disturbing the displacement field in it. Note that, in principle, the properties of the core and coating and the volume fraction c may vary from one composite cylinder to the other but they need to be constrained by the same expression (2.3). For simplicity however, we assume in this paper only one type of composite cylinders characterized by its parameters $c = (b/a)^2$, $\mu_r^{(\alpha)}$, $\mu_\theta^{(\alpha)}$. The host rod can be filled up by such composite cylinders so as to have the matrix to disappear completely, resulting thus in the CCA microgeometry. Note that since no matrix material being left now, the question of the shear modulus of the host matrix becomes irrelevant. Of course, the shear moduli of the constituents and the value of $\tilde{\mu}_m$ enter in the expressions for the warping function within the coated cylinders, but this warping vanishes at the outer boundary of those cylinders, see Section 3. Since the lateral surface of the rod can be considered to be tangent to some composite cylinder, possibly of vanishing size, we have the following important result: In a circular bar made up of the CCA microgeometry of a single type of composite cylinders, the warping function vanishes at the lateral surface bar, no matter what are the constituent properties of the core and coating.

Next, let us obtain the torsional rigidity of a bar made up of a CCA microgeometry. Again, the fact that the expression (2.5) is independent of a_0 allows us to obtain the expression for the torsional rigidity for this microgeometry just by simple summation:

$$T/\vartheta = \frac{\pi\tilde{\mu}_m R^4}{2}\left\{1 + (E/\tilde{\mu}_m)\sum_{i=1}^{\infty}(a_i/R)^4\right\}, \quad E = \mu_\theta^{(2)}c^2 + \mu_\theta^{(1)}(1 - c^2) - \tilde{\mu}_m. \tag{2.6}$$

It is important to note that this derived expression for the torsional rigidity is not based on any homogenization assumptions, and is thus valid for any size of the composite cylinders which make up the CCA. Except the trivial circumstance of a single composite cylinder which fills up the totality of the host rod, the fill up process necessitates of course the use of smaller and smaller composites cylinders in certain parts of the host rod resulting thus in an infinite number of cylinders which enter in the summation of (2.6).

The expression in (2.6) is independent on the position of the constituent composite cylinders but is dependent on their size distribution. There are two circumstances in which size distribution independence is achieved as well. The first circumstance is one in which the sizes of the composite cylinders are much smaller than the size of the host rod. Under $a_i/R = \varepsilon_i \ll 1$, it can be readily shown that

$$\operatorname*{Lim}_{\substack{N\to\infty \\ \varepsilon_i\to 0}} (T/\vartheta) = \frac{\pi\tilde{\mu}_m R^4}{2}. \tag{2.7}$$

In fact, use of the fill up condition

$$\operatorname*{Lim}_{\substack{N\to\infty \\ \varepsilon_i\to 0}} \sum_i^N \varepsilon_i^2 = 1 \tag{2.8}$$

together with

$$\sum_i^N \varepsilon_i^4 \leqslant (\mathrm{Max}\ \varepsilon_i)^2 \sum_i^N \varepsilon_i^2 \tag{2.9}$$

provides

$$\operatorname*{Lim}_{\substack{N\to\infty \\ \varepsilon_i\to 0}} \sum_i^N \varepsilon_i^4 = 0. \tag{2.10}$$

Using this result in (2.5) leads to (2.7). In this first circumstance the composite rod behaves therefore like a quasi-homogeneous cylinder with an effective shear modulus $\tilde{\mu}_m$.

The second circumstance is when $E = 0$, and consists of one constraint among the five parameters $\mu_r^{(\alpha)}$, $\mu_\theta^{(\alpha)}$, c. Under this condition, the torsional rigidity assumes a very simple form:

$$T/\vartheta = \frac{\pi R^4}{2}\{\mu_\theta^{(2)} c^2 + \mu_\theta^{(1)}(1 - c^2)\}. \tag{2.11}$$

This expression is, in fact, the torsional rigidity of one composite cylinder filling up the whole rod. It is now of interest to study some special cases of the general results stated in this section.

2.1. A CCA microgeometry in which the composite cylinders consist of an isotropic core and an isotropic coating

Each phase is now characterized by $\mu_r^{(\alpha)} = \mu_\theta^{(\alpha)} = \mu^{(\alpha)}$. In this case Eqs. (2.3) and (2.6) reduce to

$$\mu^{(m)} = \mu'_m = \frac{(\mu^{(2)}/\mu^{(1)})(1 + c) + (1 - c)}{(\mu^{(2)}/\mu^{(1)})(1 - c) + (1 + c)}\mu^{(1)}, \tag{2.12}$$

$$T/\vartheta = \frac{\pi\mu'_m R^4}{2} + \frac{\pi}{2}\sum_{i=1}^{\infty} a_i^4\{\mu^{(2)} c^2 + \mu^{(1)}(1 - c^2) - \mu'_m\}. \tag{2.13}$$

The condition $E = 0$ consists now of one constraint among the three parameters $\mu^{(1)}$, $\mu^{(2)}$, c and is given by

$$\mu^{(2)} = \mu^{(1)}\{1 + (2/c)\}, \tag{2.14}$$

whereas the torsional rigidity becomes

$$T/\vartheta = \frac{\pi R^4 \mu^{(1)}}{2}(1 + 2c). \tag{2.15}$$

Eqs. (2.12)–(2.15) reproduce those given previously by the authors, Chen et al. (2002). Note that (2.14) implies $3 \leqslant \mu^{(2)}/\mu^{(1)} < \infty$, so that the circumstance of a porous composite cylinder, for example, is ruled out.

2.2. A microgeometry consisting of non-coated cylindrically orthotropic fibers of all sizes

This is achieved under $c = 1$. The results are

$$\mu^{(m)} = \mu''_m = \sqrt{\mu_r \mu_\theta}, \qquad T/\vartheta = \frac{\pi \mu''_m R^4}{2} + \frac{\pi}{2} \sum_{i=1}^{\infty} a_i^4 \{\mu_\theta - \mu''_m\}, \tag{2.16}$$

where, since the cylinder being single phase, we have omitted any subscript on the shear moduli. Size independence in the present circumstance is achieved again under $a_i \ll R$, or in the situation of $\mu_\theta = \mu''_m$ which, in view of $(2.16)_1$ implies the trivial case of isotropy.

Finally one may ask, for given values of the parameters $\mu_r^{(\alpha)}$, $\mu_\theta^{(\alpha)}$, c, what is the size distribution which will make the torsional rigidity of the whole rod either a maximum or a minimum. We will deal with this issue in Section 4.

3. Derivation

We start by considering one coated cylinder being introduced in the host rod (Fig. 1b). We assume that the field (2.1) in host rod is undisturbed and write it in terms of the polar coordinate system located at the center of the inhomogeneity:

$$u_r^{(m)} = \vartheta a_0 x_3 \sin\theta, \qquad u_\theta^{(m)} = \vartheta x_3 r + \vartheta a_0 x_3 \cos\theta, \qquad u_3^{(m)} = 0, \tag{3.1}$$

In the coated cylinder we assume a displacement field in the form of:

$$u_r^{(\alpha)} = \vartheta a_0 x_3 \sin\theta, \quad u_\theta^{(\alpha)} = \vartheta x_3 r + \vartheta a_0 x_3 \cos\theta, \quad u_3^{(\alpha)} = \vartheta \varphi^{(\alpha)}(r,\theta) - \vartheta a_0 r \sin\theta, \qquad \alpha = 1, 2. \tag{3.2}$$

The cylindrically orthotropic coated inhomogeneity when referred to a Cartesian system behaves in a locally monoclinic manner with variable coefficients. It is shown in Appendix A that for such systems, a displacement field of the type (3.1) and (3.2) results in zero net end forces and in a twisting moment only. We have found convenient to split the warping function in (3.2) in two parts, instead of representing it by a single function as in (A.1) in Appendix A. Note that part of the field in (3.2) denoted by

$$u_r^* = \vartheta a_0 x_3 \sin\theta, \qquad u_\theta^* = \vartheta a_0 x_3 \cos\theta, \qquad u_3^* = -\vartheta a_0 r \sin\theta \tag{3.3}$$

describes a rigid body displacement.

In the described coordinate system, the stresses in the host rod and in the composite cylinder are given by

$$\begin{aligned} &\sigma_{3r}^{(m)} = \vartheta \mu^{(m)} a_0 \sin\theta, \qquad \sigma_{3\theta}^{(m)} = \vartheta \mu^{(m)} (r + a_0 \cos\theta), \\ &\sigma_{3r}^{(\alpha)} = \vartheta \mu_r^{(\alpha)} \frac{\partial \varphi^{(\alpha)}}{\partial r}, \quad \sigma_{3\theta}^{(\alpha)} = \vartheta \mu_\theta^{(\alpha)} \left(r + \frac{1}{r} \frac{\partial \varphi^{(\alpha)}}{\partial \theta} \right) \qquad \alpha = 1, 2. \end{aligned} \tag{3.4}$$

The equilibrium condition for the stresses in the composite cylinder is

$$\frac{\partial \sigma_{3r}^{(\alpha)}}{\partial r} + \frac{\sigma_{3r}^{(\alpha)}}{r} + \frac{1}{r} \frac{\partial \sigma_{3\vartheta}^{(\alpha)}}{\partial \theta} = \mu_r^{(\alpha)} \left(\frac{\partial^2 \varphi^{(\alpha)}}{\partial r^2} + \frac{1}{r} \frac{\partial \varphi^{(\alpha)}}{\partial r} \right) + \frac{1}{r^2} \mu_\theta^{(\alpha)} \frac{\partial^2 \varphi^{(\alpha)}}{\partial \theta^2} = 0, \quad \alpha = 1, 2, \tag{3.5}$$

whereas the continuity conditions for the displacement and tractions at $r = b$, $r = a$ are

$$\begin{aligned} &\varphi^{(1)}(r,\theta)\big|_{r=a} - a a_0 \sin\theta = 0, \qquad \varphi^{(1)}(r,\theta)\big|_{r=b} = \varphi^{(2)}(r,\theta)\big|_{r=b}, \\ &\mu_r^{(1)} \frac{\partial \varphi^{(1)}}{\partial r}\bigg|_{r=a} = \mu^{(m)} a_0 \sin\theta, \qquad \mu_r^{(1)} \frac{\partial \varphi^{(1)}}{\partial r}\bigg|_{r=b} = \mu_r^{(2)} \frac{\partial \varphi^{(2)}}{\partial r}\bigg|_{r=b}. \end{aligned} \tag{3.6}$$

Thus, the following form of solution for $\varphi^{(\alpha)}(r,\theta)$ is admissible:

$$\varphi^{(1)} = (Ar^{k_1} + Br^{-k_1})\sin\theta, \qquad \varphi^{(2)} = Cr^{k_2}\sin\theta, \tag{3.7}$$

where we have defined $k_\alpha = \sqrt{\mu_\theta^{(\alpha)}/\mu_r^{(\alpha)}}$. The four conditions in (3.6) provide four equations for the unknowns $\mu^{(m)}$, A, B, C. Their solution is

$$\mu^{(m)} = \tilde{\mu}_m = \sqrt{\mu_r^{(1)}\mu_\theta^{(1)}}\,\frac{g(1+c^{k_1}) + (1-c^{k_1})}{g(1-c^{k_1}) + (1+c^{k_1})}, \quad g = \frac{\sqrt{\mu_r^{(2)}\mu_\theta^{(2)}}}{\sqrt{\mu_r^{(1)}\mu_\theta^{(1)}}}, \tag{3.8}$$

$$\begin{aligned} A &= a^{1-k_1}a_0\{(\mu_G^{(1)} + \tilde{\mu}_m)/(2\mu_G^{(1)})\}, \quad \mu_G^{(1)} = \sqrt{\mu_r^{(1)}\mu_\theta^{(1)}}, \\ B &= a^{1+k_1}a_0\{(\mu_G^{(1)} - \tilde{\mu}_m)/(2\mu_G^{(1)})\}, \qquad C = Ab^{k_1-k_2} + Bb^{-k_1-k_2}. \end{aligned} \tag{3.9}$$

Next, let us obtain the torsional rigidity of the rod in which one composite cylinder has been introduced:

$$T/\vartheta = T^{(m)}/\vartheta + T^{(1)}/\vartheta + T^{(2)}/\vartheta, \tag{3.10}$$

where $T^{(m)}/\vartheta$, $T^{(1)}/\vartheta$, $T^{(2)}/\vartheta$ denote the contribution of the host rod, coating, and core of the composite cylinder, respectively. They are given by

$$T^{(\alpha)} = \int_{A_\alpha} \sigma_{3\theta}^{(\alpha)} r^2 \,\mathrm{d}r\,\mathrm{d}\theta, \tag{3.11}$$

where A_α with $\alpha = m$,1,2 denote the areas of host matrix, coating and core respectively, and the moment has been taken about the center of the inhomogeneity.

Let us first compute $T^{(m)}$. Noting that the stress field in the host matrix is the same as that existing in a circular homogeneous rod, we can readily write

$$T^{(m)}/\vartheta = \frac{\tilde{\mu}_m \pi R^4}{2} - \frac{\tilde{\mu}_m \pi a^4}{2}. \tag{3.12}$$

As to $T^{(1)}/\vartheta$ and $T^{(2)}/\vartheta$, use of (3.4) and (3.7) provides

$$T^{(\alpha)} = \int\!\!\int_{A^{(\alpha)}} \sigma_{3\theta}^{(\alpha)} r^2 \,\mathrm{d}r\,\mathrm{d}\theta = \int\!\!\int_{A^{(\alpha)}} (\vartheta\mu_\theta^{(\alpha)} r^3 + \vartheta\mu_\theta^{(\alpha)} r^2 f^{(\alpha)}\cos\theta)\,\mathrm{d}r\,\mathrm{d}\theta, \quad \alpha = 1,2, \tag{3.13}$$

where $f^{(\alpha)}$ is an expression which does not depend on the variables of integration, and drops out after the integration due to the presence of the $\cos\theta$ term. Integrating, we get

$$T^{(1)}/\vartheta = \tfrac{1}{2}\pi\mu_\theta^{(1)} a^4(1-c^2), \qquad T^{(2)}/\vartheta = \tfrac{1}{2}\pi\mu_\theta^{(2)} a^4 c^2. \tag{3.14}$$

Finally use of (3.12) and (3.14) in (3.10) produces the desired result (2.5). We remark here that if we take moments about the center of the host bar instead of the center of the inhomogeneity, the algebra becomes very complicated. Yet, just to verify our results we have carried out this second option as well (using MAPLE software), and recovered of course the same equation.

We now turn to a discussion of an interesting observation on the derived expression for $\tilde{\mu}_m$ in (3.8). It turns out that this expression is the same as that derived by Hashin (1990) for the effective anti-plane shear modulus of CCA made up of cylindrically orthotropic constituents (his Eq. (28)). It will be now shown that this coincidence is due to an underlying correspondence between the following two problems. *Problem I*:

The anti-plane problem of a neutral cylindrical inhomogeneity of a circular cross section in a circular rod (here "a neutral inhomogeneity" is defined as one which leaves the anti-plane field in the host rod undisturbed). *Problem II*: The Saint-Venant torsion problem of a partially neutral inhomogeneity described at the start of this section. It is important to mention that this correspondence is valid only for the case of a circular inhomogeniety in the host rod and will not hold, for example, if the inhomogeneity has an elliptical cross section. Thus, this is not a universal analogy of the well-known types between the Saint-Venant torsion and other physical phenomena. The existence of this correspondence for the case of a composite cylinder whose constituents were isotropic was established in Chen et al. (2002). Here we show that this correspondence is valid for the more general circumstance in which the cylindrical inhomogeneity of circular cross section is cylindrically orthotropic and inhomogeneous (a continuously graded dependence with r) or multilayered with constant material properties in each layer. In fact, it is sufficient to show the correspondence for the graded case.

We first state the governing equations in Problem I. Consider an isotropic and homogeneous cylindrical bar of circular cross section of radius R. Let its shear modulus be denoted by $\mu^{(m)}$. Define a Cartesian coordinate system (x_1, x_2, x_3) centered at one end of the bar and let the axis of the bar lie along $x_1 = -a_0$, $x_2 = 0$. Subject now the boundary S of the bar to a displacement field in the form:

$$u_1^{(m)}(S) = u_2^{(m)}(S) = 0, \qquad u_3^{(m)}(S) = \gamma^0 x_2, \tag{3.15}$$

where $\gamma^0 = 2\varepsilon_{23}^0$ is a constant shear strain. The following displacement and stress fields prevail in the bar:

$$u_1^{(m)}(x_1,x_2,x_3) = u_2^{(m)}(x_1,x_2,x_3) = 0, \qquad u_3^{(m)}(x_1,x_2,x_3) = \gamma^0 x_2, \qquad \sigma_{32}^{(m)}(x_1,x_2,x_3) = \mu^{(m)}\gamma^0. \tag{3.16}$$

Now let us introduce a cylindrical inhomogeneity of a circular cross section of radius a in the host bar, and let its axis lie along $x_1 = 0$, $x_2 = 0$. Locate a cylindrical coordinate system (r, θ, x_3) whose x_3-axis coincides with that of the Cartesian system. Let the inhomogeneity be cylindrically orthotropic and exhibit a graded dependence on r:

$$\sigma_{3r}^{(f)} = 2\mu_r^{(f)}(r)\varepsilon_{3r}^{(f)}, \qquad \sigma_{3\theta}^{(f)} = 2\mu_\theta^{(f)}(r)\varepsilon_{3\theta}^{(f)}, \tag{3.17}$$

where we have denoted the quantities pertaining to the inhomogeneity by "f". Demand now the field outside the inhomogeneity to remain the same after its introduction in the host bar, and ask if there is the specific value of $\mu^{(m)}$ which makes this possible.

Assume the following displacement field in the inhomogeneity:

$$u_1^{(f)}(x_1,x_2,x_3) = u_2^{(f)}(x_1,x_2,x_3) = 0, \qquad u_3^{(f)}(x_1,x_2,x_3) = \psi^{(f)}(x_1,x_2) \tag{3.18}$$

which results in the following stresses:

$$\boldsymbol{\sigma} = (\sigma_{3r}^{(f)}, \sigma_{3\theta}^{(f)}) = \left(\mu_r^{(f)}(r)\frac{\partial\psi^{(f)}}{\partial r}, \mu_\theta^{(f)}(r)\frac{1}{r}\frac{\partial\psi^{(f)}}{\partial\theta}\right). \tag{3.19}$$

The equilibrium equation for the stresses become

$$\frac{\partial\sigma_{3r}^{(f)}}{\partial r} + \frac{\sigma_{3r}^{(f)}}{r} + \frac{1}{r}\frac{\partial\sigma_{3\theta}^{(f)}}{\partial\theta} = \frac{\partial}{\partial r}\left(\mu_r^{(f)}(r)\frac{\partial\psi^{(f)}}{\partial r}\right) + \frac{1}{r}\mu_r^{(f)}\frac{\partial\psi^{(f)}}{\partial r} + \frac{1}{r}\frac{\partial}{\partial\theta}\left(\mu_\theta^{(f)}(r)\frac{1}{r}\frac{\partial\psi^{(f)}}{\partial\theta}\right) = 0 \tag{3.20}$$

and are accompanied by the following boundary conditions:

$$\psi^{(f)}\big|_{r=a} = \gamma^0 a\sin\theta, \qquad \mu_r^{(f)}(a)\frac{\partial\psi^{(f)}}{\partial r}\bigg|_{r=a} = \gamma^0\mu^{(m)}\sin\theta. \tag{3.21}$$

Eqs. (3.18) and (3.20) govern the problem of a neutral circular inhomogeneity in the anti-plane elasticity context. For the case in which the inhomogeneity is made up of a core and coating which are cylindrically orthotropic, Hashin (1990) has shown that the sought value of $\mu^{(m)}$ is given by (2.3). [1]

Next, let us turn to Problem II. Consider the homogeneous circular bar of shear modulus $\mu^{(m)}$ with the cylindrically orthotropic inhomogeneity being introduced in it, and subject it to Saint-Venant torsion. We ask whether there is a specific value of the shear modulus $\mu^{(m)}$ so that the displacement field in the host bar remains unchanged after the introduction of the inhomogeneity. This undisturbed displacement field in the host bar is given by (3.1), and results in a stress field given at the first line of (3.4). Along the lines of Section 3, we look for a displacement field in the inhomogeneity described by

$$u_r^{(f)} = \vartheta a_0 x_3 \sin\theta, \quad u_\theta^{(f)} = \vartheta x_3 r + \vartheta a_0 x_3 \cos\theta, \quad u_3^{(f)} = \vartheta\varphi^{(f)}(r,\theta) - \vartheta a_0 r \sin\theta \tag{3.22}$$

which results in a stress field:

$$\sigma_{3r}^{(f)} = \vartheta\mu_r^{(f)}(r)\frac{\partial\varphi^{(f)}}{\partial r}, \qquad \sigma_{3\theta}^{(f)} = \vartheta\mu_\theta^{(f)}(r)\left(r + \frac{1}{r}\frac{\partial\varphi^{(f)}}{\partial\theta}\right). \tag{3.23}$$

Fulfillment of the equilibrium condition requires

$$\frac{\partial\sigma_{3r}^{(f)}}{\partial r} + \frac{\sigma_{3r}^{(f)}}{r} + \frac{1}{r}\frac{\partial\sigma_{3\vartheta}^{(f)}}{\partial\theta} = \frac{\partial}{\partial r}\left(\mu_r^{(f)}(r)\frac{\partial\varphi^{(f)}}{\partial r}\right) + \frac{1}{r}\mu_r^{(f)}\frac{\partial\varphi^{(f)}}{\partial r} + \frac{1}{r}\frac{\partial}{\partial\theta}\left(\mu_\theta^{(f)}(r)\frac{1}{r}\frac{\partial\varphi^{(f)}}{\partial\theta}\right) = 0 \tag{3.24}$$

whereas the boundary conditions at $r = a$ necessitates

$$\varphi^{(f)}\big|_{r=a} = a_0 a \sin\theta, \qquad \mu_r^{(f)}(a)\frac{\partial\varphi^{(f)}}{\partial r}\bigg|_{r=a} = a_0\mu^{(m)}\sin\theta. \tag{3.25}$$

The correspondence between $\psi^{(f)}$ and $\varphi^{(f)}$ in Problem I ((3.20), (3.21)) and Problem II ((3.24), (3.25)) becomes now obvious under $\psi^{(f)} \Longleftrightarrow \varphi^{(f)}$ and $\gamma^0 \Longleftrightarrow a_0$ (note that the dimension of $\psi^{(f)}$ is [length] whereas the dimension of $\varphi^{(f)}$ is [length]^2). Clearly, the above proof encompasses the case in which the inhomogeneity is multilayered as well.

It is important to clarify here why the above correspondence fails to hold if the cross section of the inhomogeneity is not circular. Although this feature was already indicated in Chen et al. (2002), for the sake of completeness, we present here a slightly different and shorter explanation. Consider for example, a two phase inhomogeneity in which both the core and coating are isotropic. Let the inhomogeneity be of arbitrary shape and denote the inner and outer boundary of the coating by the closed curves Γ_i and Γ_o respectively. Let the outer unit normal to these curves be defined by **m** and **n**. Locate the inhomogeneity at an arbitrary position in the host bar, and define a Cartesian coordinate system (x_1, x_2, x_3) at an arbitrary location, say in the core, see Fig. 2. Let the axis of the host bar be at $x_1 = -a_0$, $x_2 = 0$. The problem of neutral inhomogeneities in the anti-plane elasticity is defined by (this problem has been studied in depth in Milton and Serkov, 2001)

[1] Note that the anti-plane displacement field used by Hashin allows also for a u_2 component which is linear in x_3: $u_1^{(m)} = 0$, $u_2^{(m)} = (2\varepsilon_{23}^0)x_3$, $u_3^{(m)} = (2\varepsilon_{23}^0)x_2$, but yet it leads to the same $\tilde{\mu}_m$.

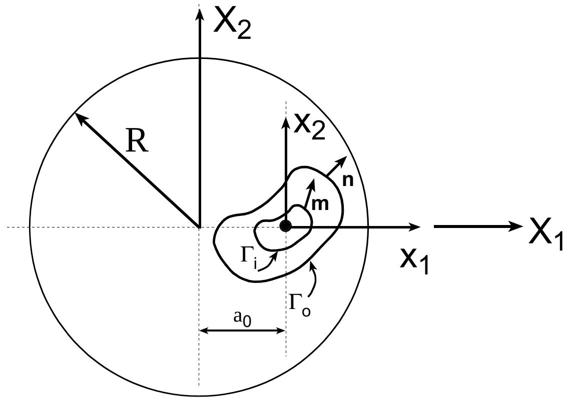

Fig. 2. A coated inhomogeneity of arbitrary shape in a circular bar.

$$
\begin{aligned}
&u_1^{(m)} = u_2^{(m)} = 0, \qquad u_3^{(m)}(x_1,x_2) = \gamma^0 x_2,\\
&u_1^{(\alpha)} = u_2^{(\alpha)} = 0, \quad u_3^{(\alpha)}(x_1,x_2) = \chi^{(\alpha)}(x_1,x_2), \quad \nabla^2\psi^{(\alpha)} = 0, \qquad \alpha = 1,2,\\
&\psi^{(1)}\big|_{\Gamma_\mathrm{o}} = (\gamma^0 x_2)\big|_{\Gamma_\mathrm{o}}, \qquad \mu^{(1)}\frac{\partial\psi^{(1)}}{\partial n}\bigg|_{\Gamma_\mathrm{o}} = \mu^{(m)}\gamma^0 n_2,\\
&\psi^{(2)}\big|_{\Gamma_\mathrm{i}} = \psi^{(1)}\big|_{\Gamma_\mathrm{i}}, \qquad \mu^{(2)}\frac{\partial\psi^{(2)}}{\partial m}\bigg|_{\Gamma_\mathrm{i}} = \mu^{(1)}\frac{\partial\psi^{(1)}}{\partial m}\bigg|_{\Gamma_\mathrm{i}}.
\end{aligned}
\tag{3.26}
$$

The Saint-Venant torsion problem of the same configuration, on the other hand, is governed by (see Chen et al., 2002)

$$
\begin{aligned}
&u_1^{(m)} = -\vartheta x_3 x_2, \quad u_2^{(m)} = \vartheta x_3(x_1 + a_0), \quad u_3^{(m)} = 0,\\
&u_1^{(\alpha)} = -\vartheta x_3 x_2, \quad u_2^{(\alpha)} = \vartheta x_3(x_1 + a_0), \quad u_3^{(\alpha)} = \vartheta\varphi^{(\alpha)} - \vartheta x_2 a_0, \quad \nabla^2\varphi^{(\alpha)} = 0, \qquad \alpha = 1,2,\\
&\varphi^{(1)}\big|_{\Gamma_\mathrm{o}} = (x_2 a_0)\big|_{\Gamma_\mathrm{o}}, \quad \mu^{(1)}\left(\frac{\partial\varphi^{(1)}}{\partial n} + \mathbf{v}\cdot\mathbf{n}\right)\bigg|_{\Gamma_\mathrm{o}} = \mu^{(m)}(a_0 n_2 + \mathbf{v}\cdot\mathbf{n})\big|_{\Gamma_\mathrm{o}},\\
&\varphi^{(2)}\big|_{\Gamma_\mathrm{i}} = \varphi^{(1)}\big|_{\Gamma_\mathrm{i}}, \quad \mu^{(2)}\left(\frac{\partial\varphi^{(2)}}{\partial m} + \mathbf{v}\cdot\mathbf{m}\right)\bigg|_{\Gamma_\mathrm{i}} = \mu^{(1)}\left(\frac{\partial\varphi^{(1)}}{\partial m} + \mathbf{v}\cdot\mathbf{m}\right)\bigg|_{\Gamma_\mathrm{i}}, \qquad \alpha = 1,2,
\end{aligned}
\tag{3.27}
$$

where $\mathbf{v} = -x_2\mathbf{i} + x_1\mathbf{j}$. Comparing (3.26) and (3.27), it become obvious that a correspondence of the nature $\psi^{(f)} \Longleftrightarrow \varphi^{(f)}$ and $\gamma^0 \Longleftrightarrow a_0$ is possible only if $\mathbf{v}\cdot\mathbf{n} = \mathbf{v}\cdot\mathbf{m} = 0$. The only instance when this is realized is that in which the boundaries Γ_i and Γ_o are circular.

4. Maximum and minimum torsional rigidities of the CCA microgeometry

As seen above, the derived torsional rigidity (2.6) for a circular bar made of the CCA microgeometry is dependent on the sizes of the composite cylinders which fill it up. We now ask the following question: "among all the possible size distributions of composite cylinders (with given properties and volume fractions of the core and coating) which one makes the torsional rigidity of the bar a maximum, and which one makes it a minimum?" It turns out that an inequality analysis allows to provide an answer to this question in the following two circumstances:

$$\text{(a)}\quad \mu_\theta^{(1)} \geqslant \mu_r^{(1)}, \qquad \mu_\theta^{(2)} \geqslant \mu_r^{(2)}, \qquad \mu_\theta^{(1)}\mu_r^{(1)} \geqslant \mu_\theta^{(2)}\mu_r^{(2)}. \tag{4.1}$$

$$\text{(b)}\quad \mu_\theta^{(1)} \leqslant \mu_r^{(1)}, \qquad \mu_\theta^{(2)} \leqslant \mu_r^{(2)}, \qquad \mu_\theta^{(1)}\mu_r^{(1)} \leqslant \mu_\theta^{(2)}\mu_r^{(2)}. \tag{4.2}$$

4.1. The case of $\mu_\theta^{(1)} \geqslant \mu_r^{(1)}$, $\mu_\theta^{(2)} \geqslant \mu_r^{(2)}$, $\mu_\theta^{(1)}\mu_r^{(1)} \geqslant \mu_\theta^{(2)}\mu_r^{(2)}$

The maximum torsional rigidity in this case is achieved by a single inhomogeneity which fills up the whole bar and is given by

$$(T/\vartheta)_{\text{Max}} = \frac{\pi R^4}{2}\{\mu_\theta^{(2)}c^2 + \mu_\theta^{(1)}(1-c^2)\}. \tag{4.3}$$

We need to show that

$$\frac{\pi\tilde{\mu}_m R^4}{2}\left\{1 + \frac{1}{\tilde{\mu}_m}(\mu_\theta^{(2)}c^2 + \mu_\theta^{(1)}(1-c^2) - \tilde{\mu}_m)\sum_{i=1}^{N}(a_i/R)^4\right\} \leqslant \frac{\pi R^4}{2}\{\mu_\theta^{(2)}c^2 + \mu_\theta^{(1)}(1-c^2)\}, \tag{4.4}$$

which can be cast in the form

$$(\mu_\theta^{(2)}c^2 + \mu_\theta^{(1)}(1-c^2) - \tilde{\mu}_m)\left(\sum_{i=1}^{N} a_i^4 - R^4\right) \leqslant 0. \tag{4.5}$$

First, we note that $(\sum_{i=1}^{N} a_i^4 - R^4) \leqslant 0$ which follows by considering the fill up condition:

$$\pi\sum_{i}^{N} a_i^2 = \pi R^2 \Rightarrow \sum_{i=1}^{N} a_i^4 = R^4 - 2\sum_{i}^{N}\sum_{j\neq i}^{N} a_i^2 a_j^2 \leqslant R^4. \tag{4.6}$$

Next, we need to establish

$$(\mu_\theta^{(2)}c^2 + \mu_\theta^{(1)}(1-c^2) - \tilde{\mu}_m) \geqslant 0, \tag{4.7}$$

which can be cast in the following form after invoking the definition of $\tilde{\mu}_m$ in (2.3):

$$\frac{1}{\mu_G^{(1)}}\{\mu_\theta^{(2)}c^2 + \mu_\theta^{(1)}(1-c^2)\} \geqslant \frac{(g+1) + c^{k_1}(g-1)}{(g+1) + c^{k_1}(1-g)}, \tag{4.8}$$

where all the parameters have been defined in (3.8) and (3.9). Proving the inequality (4.8) necessitates a few steps. The first step is to prove that

$$g\tilde{c}^2 + (1-\tilde{c}^2) \geqslant \frac{(g+1) + \tilde{c}(g-1)}{(g+1) + \tilde{c}(1-g)}, \tag{4.9}$$

where we have introduced $\tilde{c} = c^{k_1}$. Some manipulation shows that the correctness of (4.9) depends on the validity of

$$\tilde{c}^2(1-g)^2 - \tilde{c}(g^2-1) + 2(g-1) \leqslant 0. \tag{4.10}$$

In view of the orders of magnitudes of the parameters prevailing in this case, one has $0 \leqslant \tilde{c} \leqslant 1$ and $0 \leqslant g \leqslant 1$. It can be readily established that under these conditions (4.10) is valid, and thus (4.9) as well.

In the next step, we make note of the fact that if the following inequality

$$\frac{1}{\mu_G^{(1)}}\{\mu_\theta^{(2)}c^2+\mu_\theta^{(1)}(1-c^2)\}\geqslant g\tilde{c}^2+(1-\tilde{c}^2) \tag{4.11}$$

could be proved, then use of (4.11) and (4.9) allows the establishment of the desired main inequality (4.8). The proof of (4.11) is achieved by casting it, after some manipulation, in the form:

$$\mu_\theta^{(1)}F(C;k_1)+\mu_\theta^{(2)}G(C;k_1,k_2)\geqslant 0, \tag{4.12}$$

where $C=c^2$, $k_\alpha=\sqrt{\mu_\theta^{(\alpha)}/\mu_r^{(\alpha)}}\geqslant 1$, $\alpha=1,2$ (in view of (4.1)), and $F(C;k_1)$, $G(C;k_1,k_2)$ have been defined as

$$F(C;k_1)=1-C+\frac{1}{k_1}(C^{k_1}-1),\qquad G(C;k_1,k_2)=C-\frac{1}{k_2}C^{k_1}. \tag{4.13}$$

Note that $F(0;k_1)=1-\frac{1}{k_1}\geqslant 0$, $F(1;k_1)=0$ and $\mathrm{d}F/\mathrm{d}C=C^{k_1-1}-1\leqslant 0$; thus $F(C;k_1)\geqslant 0$ in the range of interest $0\leqslant C\leqslant 1$ of C. The positiveness of $G(C;k_1,k_2)$ on the other hand is readily transparent. This establishes the truth of (4.12), (4.11) and thus of the desired inequality (4.8).

Let us now prove that the minimum torsional rigidity is achieved by a CCA microgeometry with $a_i\ll R$ and is given by

$$(T/\vartheta)_{\mathrm{Min}}=\frac{\pi\tilde{\mu}_mR^4}{2}.$$

To establish this result we need to have

$$\frac{\pi\tilde{\mu}_mR^4}{2}\left\{1+\frac{1}{\tilde{\mu}_m}(\mu_\theta^{(2)}c^2+\mu_\theta^{(1)}(1-c^2)-\tilde{\mu}_m)\sum_{i=1}^{N}(a_i/R)^4\right\}\geqslant\frac{\pi\tilde{\mu}_mR^4}{2}, \tag{4.14}$$

which is seen to be valid in view of the already established inequality (4.7).

4.2. The case of $\mu_\theta^{(1)}\leqslant\mu_r^{(1)}$, $\mu_\theta^{(2)}\leqslant\mu_r^{(2)}$, $\mu_\theta^{(1)}\mu_r^{(1)}\leqslant\mu_\theta^{(2)}\mu_r^{(2)}$

The maximum torsional rigidity is now achieved by the CCA microgeometry with $a_i\ll R$, and is given by

$$(T/\vartheta)_{\mathrm{Max}}=\frac{\pi\tilde{\mu}_mR^4}{2}.$$

This necessitates that

$$(\mu_\theta^{(2)}c^2+\mu_\theta^{(1)}(1-c^2)-\tilde{\mu}_m)\leqslant 0 \tag{4.15}$$

or

$$\frac{1}{\mu_G^{(1)}}\{\mu_\theta^{(2)}c^2+\mu_\theta^{(1)}(1-c^2)\}\leqslant\frac{(g+1)+c^{k_1}(g-1)}{(g+1)+c^{k_1}(1-g)}. \tag{4.16}$$

Similar to the development between (4.8) and (4.13), the validity of (4.16) follows from the correctness of the following inequalities under the conditions of (4.2):

$$\tilde{c}^2(1-g)^2-\tilde{c}(g^2-1)+2(g-1)\geqslant 0, \tag{4.17}$$

$$\frac{1}{\mu_G^{(1)}}\{\mu_\theta^{(2)}c^2+\mu_\theta^{(1)}(1-c^2)\}\leqslant g\tilde{c}^2+(1-\tilde{c}^2), \tag{4.18}$$

$$\mu_\theta^{(1)}F(C;k_1)+\mu_\theta^{(2)}G(C;k_1,k_2)\leqslant 0. \tag{4.19}$$

The minimum torsional rigidity under the conditions of (4.2) occurs by a single inhomogeneity filling up the whole rod, and is given by the same expression as in (4.3). The proof is based on the validity of $(\sum_{i=1}^{N} a_i^4 - R^4) \leqslant 0$ and (4.15).

Finally, we make note of the fact that the conditions in (4.1) and (4.2) cover as special cases the circumstances in which the cylinders which fill up the host rod are single phase with cylindrical orthotropy, and two phase with isotropic constituents in the core and coating.

Appendix A

Consider a cylindrical bar of arbitrary cross section containing a cylindrical inhomogeneity. Let the lateral boundary of the bar be denoted by S and the interface between the inhomogeneity and the host bar be denoted by Γ. We will assume that Γ is closed. Define a Cartesian coordinate system located at the end of the bar and let x_3 denote the axial direction. We assume that both the inhomogeneity and the host matrix are locally monoclinic with (x_1, x_2) denoting the plane of reflectional symmetry. Under applied end torques we assume that the displacement field in the bar are given by

$$u_1^{(r)} = -\vartheta x_2 x_3, \quad u_2^{(r)} = \vartheta x_1 x_3, \quad u_3^{(r)} = \vartheta \chi^{(r)}(x_1, x_2), \qquad r = f, m, \tag{A.1}$$

where ϑ is the angle of twist per unit length and $r = f$ denotes the inhomogeneity whereas $r = m$ denotes the matrix. The resulting strains and stresses are

$$\begin{aligned}
&2\varepsilon_{13}^{(r)} = \vartheta\left(\frac{\partial \chi^{(r)}}{\partial x_1} - x_2\right), \qquad 2\varepsilon_{23}^{(r)} = \vartheta\left(\frac{\partial \chi^{(\alpha)}}{\partial x_2} + x_1\right),\\
&\varepsilon_{12}^{(r)} = \varepsilon_{11}^{(r)} = \varepsilon_{22}^{(r)} = \varepsilon_{33}^{(r)} = 0,\\
&\sigma_{23}^{(r)} = \vartheta\left\{C_{44}^{(r)}(x_1, x_2)\left(\frac{\partial \chi^{(r)}}{\partial x_2} + x_1\right) + C_{45}^{(r)}(x_1, x_2)\left(\frac{\partial \chi^{(r)}}{\partial x_1} - x_2\right)\right\},\\
&\sigma_{13}^{(r)} = \vartheta\left\{C_{45}^{(r)}(x_1, x_2)\left(\frac{\partial \chi^{(r)}}{\partial x_2} + x_1\right) + C_{55}^{(r)}(x_1, x_2)\left(\frac{\partial \chi^{(r)}}{\partial x_1} - x_2\right)\right\},\\
&\sigma_{12}^{(r)} = \sigma_{11}^{(r)} = \sigma_{22}^{(r)} = \sigma_{33}^{(r)} = 0,
\end{aligned} \tag{A.2}$$

where $C_{44}^{(r)}(x_1, x_2)$, $C_{45}^{(r)}(x_1, x_2)$, $C_{55}^{(r)}(x_1, x_2)$ are the monoclinic moduli active in the present deformation conditions, and are assumed to vary in x_1 and x_2.

The equilibrium condition of the stresses reads

$$\frac{\partial}{\partial x_1}\sigma_{31}^{(r)} + \frac{\partial}{\partial x_2}\sigma_{32}^{(r)} = 0, \tag{A.3}$$

whereas the vanishing condition of the tractions at S, and their continuity at Γ are

$$(\sigma_{31}^{(m)} n_1 + \sigma_{32}^{(m)} n_2)_S = 0, \qquad (\sigma_{31}^{(f)} m_1 + \sigma_{32}^{(f)} m_2)_\Gamma = (\sigma_{31}^{(m)} m_1 + \sigma_{32}^{(m)} m_2)_\Gamma, \tag{A.4}$$

where $\mathbf{n} = (n_1, n_2)$ denotes the outer unit normal to S and $\mathbf{m} = (m_1, m_2)$ is the unit normal to Γ pointing from the inhomogeneity into the matrix. The continuity of the displacements at Γ demand:

$$(\chi^{(f)})_\Gamma = (\chi^{(m)})_\Gamma. \tag{A.5}$$

It can now be shown that with $\chi^{(r)}$ being characterized by (A.2)–(A.5), the displacement field in (A.1) results in vanishing forces at the ends of the bar and produces a torsional moment only. In view of the stress field as described in (A.2), we need to establish

$$\int_A \sigma_{31}\,\mathrm{d}A = 0, \qquad \int_A \sigma_{32}\,\mathrm{d}A = 0, \tag{A.6}$$

where A denotes the cross section of the bar. Let us consider the first integral

$$\int_A \sigma_{31}\,\mathrm{d}A = \vartheta \sum_{r=f,m} \int_{A_r} \sigma_{31}^{(r)}\,\mathrm{d}A_r = \vartheta \sum_{r=f,m} \int_{A_r} \left\{ \frac{\partial}{\partial x_1}(x_1\sigma_{31}^{(r)}) + \frac{\partial}{\partial x_2}(x_1\sigma_{32}^{(r)}) \right\} \mathrm{d}A_r, \tag{A.7}$$

where the second equality follows by invoking (A.3). Application of the divergence theorem to (A.7) and the consecutive use of the boundary and interface conditions (A.4) readily shows that the net force in the x_1-direction is zero. The vanishing of the second integral in (A.6) is shown in the same manner.

Since there are no transverse net forces at the ends of the bar, the resultant torques there can be computed by taking moments of the stresses about any point of the cross section. Furthermore it can also be established that the choice of the origin of the Cartesian system does not affect the stresses and affects the displacement fields only within a rigid body motion. To see this property, let us chose a coordinate system (x_1', x_2', x_3') related to the first one by

$$x_1 = x_1' + a_1, \quad x_2 = x_2' + a_2, \quad x_3 = x_3'. \tag{A.8}$$

The counterpart to the displacement fields in (A.1) are

$$u_1'^{(r)} = -\vartheta x_2' x_3', \quad u_2'^{(r)} = \vartheta x_1' x_3', \quad u_3'^{(r)} = \vartheta \chi'^{(r)} \tag{A.9}$$

or

$$u_1'^{(r)} = -\vartheta (x_2 - a_2) x_3, \quad u_2'^{(r)} = \vartheta (x_1 - a_1) x_3, \quad u_3'^{(r)} = \vartheta \chi'^{(r)} \tag{A.10}$$

which results in the following non-vanishing strains and stresses:

$$\begin{aligned}
&2\varepsilon_{13}'^{(r)} = \vartheta\left(\frac{\partial \chi'^{(r)}}{\partial x_1} - x_2 + a_2\right), \qquad 2\varepsilon_{23}'^{(r)} = \vartheta\left(\frac{\partial \chi'^{(r)}}{\partial x_2} + x_1 - a_1\right),\\
&\sigma_{23}'^{(r)} = \vartheta\left\{ C_{44}^{(r)}(x_1,x_2)\left(\frac{\partial \chi'^{(r)}}{\partial x_2} + x_1 - a_1\right) + C_{45}^{(r)}(x_1,x_2)\left(\frac{\partial \chi'^{(r)}}{\partial x_1} - x_2 + a_2\right)\right\},\\
&\sigma_{13}'^{(r)} = \vartheta\left\{ C_{45}^{(r)}(x_1,x_2)\left(\frac{\partial \chi'^{(r)}}{\partial x_2} + x_1 - a_1\right) + C_{55}^{(r)}(x_1,x_2)\left(\frac{\partial \chi'^{(r)}}{\partial x_1} - x_2 + a_2\right)\right\}.
\end{aligned} \tag{A.11}$$

The stresses $\sigma_{31}'^{(r)}$ and $\sigma_{32}'^{(r)}$ need to satisfy

$$(\sigma_{31}'^{(m)} n_1 + \sigma_{32}'^{(m)} n_2)_S = 0, \qquad (\sigma_{31}'^{(f)} m_1 + \sigma_{32}'^{(f)} m_2)_\Gamma = (\sigma_{31}'^{(m)} m_1 + \sigma_{32}'^{(m)} m_2)_\Gamma, \tag{A.12}$$

whereas the warping function obeys

$$(\chi'^{(f)})_\Gamma = (\chi'^{(m)})_\Gamma. \tag{A.13}$$

It is now easy to check that if $\chi^{(r)}$ is a solution of (A.2)–(A.5), then $\chi'^{(r)}$ defined by

$$\chi'^{(r)} = \chi^{(r)} - a_2 x_1 + a_1 x_2 \tag{A.14}$$

will be a solution of (A.11)–(A.13). This readily establishes that the stresses in (A.2) and (A.11) are the same and the displacement fields differ from each other only within a rigid body motion.

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